P&S Problem 2.1: Classical electromagnetism

2.1 (a) Find the Maxwell equations from the action.

1. Start with the classical electromagnetic action, no sources:

   $S = \int dx^4 \, \frac{1}{4} \, (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu})$.

2. Take the variation of the action:

   $\delta S = \int dx^4 \, \frac{\partial \frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu})}{\partial A^\mu} \delta A^\mu + \frac{\partial \frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu})}{\partial A^{\mu,\nu}} \delta A^{\mu,\nu}$.

   The first term is zero.

3. Apply the chain rule to $\delta A^{\mu,\nu}$:

   $\delta S = \int dx^4 \, (\frac{\partial \frac{1}{4} (A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu})}{\partial A^{\mu,\nu}} \delta A^\mu)^{,\nu} - \frac{\partial \frac{1}{4} ((A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}))^{,\nu}}{\partial A^{\mu,\nu}} \delta A^\mu$.

   There is a theorem from Gauss that says the first term above is zero.

4. The action will be an extremum if $\delta S = 0$. This will always be the case if the integrand is zero:

   $\frac{\partial \, ((A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}))^{,\nu}}{\partial A^{\mu,\nu}} = (A_{\mu,\nu} - A_{\nu,\mu})^{,\nu} = 0.$

5. Write out $(A_{\mu,\nu} - A_{\nu,\mu})$:

   $$\begin{pmatrix} 0 & -\frac{\partial A_x}{\partial t} - c\frac{\partial \phi}{\partial x} & -\frac{\partial A_y}{\partial t} - c\frac{\partial \phi}{\partial y} & -\frac{\partial A_z}{\partial t} - c\frac{\partial \phi}{\partial z} \\ c\frac{\partial \phi}{\partial x} + \frac{\partial A_x}{\partial t} & 0 & -c\frac{\partial A_y}{\partial x} + c\frac{\partial A_x}{\partial y} & -c\frac{\partial A_z}{\partial x} + c\frac{\partial A_x}{\partial z} \\ c\frac{\partial \phi}{\partial y} + \frac{\partial A_y}{\partial t} & -c\frac{\partial A_x}{\partial y} + c\frac{\partial A_y}{\partial x} & 0 & -c\frac{\partial A_z}{\partial y} + c\frac{\partial A_y}{\partial z} \\ c\frac{\partial \phi}{\partial z} + \frac{\partial A_z}{\partial t} & -c\frac{\partial A_x}{\partial z} + c\frac{\partial A_z}{\partial x} & -c\frac{\partial A_y}{\partial z} + c\frac{\partial A_z}{\partial y} & 0 \end{pmatrix}.$$

6. Contract $F_{\mu\nu}$ with the contravariant derivative $^{,\nu}$. Apply each derivative along a row with negative signs for the spatial ones, and sum up the columns to get the four source Maxwell equations:

   $$-\frac{\partial}{\partial x}(c\frac{\partial \phi}{\partial x} + \frac{\partial A_x}{\partial t}) - \frac{\partial}{\partial y}(c\frac{\partial \phi}{\partial y} + \frac{\partial A_y}{\partial t}) - \frac{\partial}{\partial z}(c\frac{\partial \phi}{\partial z} + \frac{\partial A_z}{\partial t}) = \vec{\nabla} \cdot \vec{E} = 0$$

   $$\frac{\partial}{c\partial t}(-\frac{\partial A_x}{\partial t} - c\frac{\partial \phi}{\partial x}) - \frac{\partial}{\partial y}(-c\frac{\partial A_x}{\partial y} + c\frac{\partial A_y}{\partial x}) - \frac{\partial}{\partial z}(-c\frac{\partial A_x}{\partial z} + c\frac{\partial A_z}{\partial x}) \Rightarrow \frac{\partial \vec{E}}{c\partial t} - \vec{\nabla} \times \vec{B} = 0 \ .$$

   Theses are Gauss and Ampere's laws.

   The homogeneous equations are vector identities.

2.2 The energy and momentum densities.

First calculate the energy density from the Hamiltonian density.

1. Start from the Lagrange density:

   $$\mathfrak{L} = -\frac{1}{4}\,(A_{\mu,\nu} - A_{\nu,\mu})(A^{\mu,\nu} - A^{\nu,\mu}).$$

2. Write out all the components:

   $$\mathfrak{L} = \frac{1}{2}((\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2 + (\frac{\partial A_x}{c\partial t})^2 - (\frac{\partial A_x}{\partial y})^2 - (\frac{\partial A_x}{\partial z})^2 \qquad \text{quadratics}$$

   $$+ (\frac{\partial A_y}{c\partial t})^2 - (\frac{\partial A_y}{\partial x})^2 - (\frac{\partial A_y}{\partial z})^2 + (\frac{\partial A_z}{c\partial t})^2 - (\frac{\partial A_z}{\partial x})^2 - (c\frac{\partial A_z}{\partial y})^2)$$

   $$+ \frac{\partial\phi}{\partial x}\frac{\partial A_x}{c\partial t} + \frac{\partial\phi}{\partial y}\frac{\partial A_y}{c\partial t} + \frac{\partial\phi}{\partial z}\frac{\partial A_z}{c\partial t} + \frac{\partial A_x}{\partial y}\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial z}\frac{\partial A_z}{\partial x} + \frac{\partial A_y}{\partial z}\frac{\partial A_z}{\partial y} \quad . \qquad \text{cross terms}$$

3. Calculate the connonical momentum density conjugate to A^μ:

   $$\pi_\lambda = \frac{\partial\mathfrak{L}}{\partial(\frac{\partial A^\lambda}{c\partial t})} = (0, \frac{\partial A_x}{c\partial t} + \frac{\partial\phi}{\partial x}, \frac{\partial A_y}{c\partial t} + \frac{\partial\phi}{\partial y}, \frac{\partial A_z}{c\partial t} + \frac{\partial\phi}{\partial z}) \quad .$$

4. Calculate the Hamiltonian density which is the 00 component of the stress energy tensor, minus terms to make the stress energy tensor symmetric:

   $$h = \pi_\lambda\frac{\partial A^\lambda}{c\partial t} - \mathfrak{L} - \vec{\nabla}\cdot\phi\vec{E}$$

   $$= \frac{1}{2}(\frac{\partial A_x}{c\partial t})^2 + \frac{\partial\phi}{\partial x}\frac{\partial A_x}{c\partial t} + \frac{1}{2}(\frac{\partial A_y}{c\partial t})^2 + \frac{\partial\phi}{\partial y}\frac{\partial A_y}{c\partial t} + \frac{1}{2}(\frac{\partial A_z}{c\partial t})^2 + \frac{\partial\phi}{\partial z}\frac{\partial A_z}{c\partial t} \qquad \pi_\lambda\frac{\partial A^\lambda}{c\partial t}$$

   $$+ \frac{1}{2}(- (\frac{\partial\phi}{\partial x})^2 - (\frac{\partial\phi}{\partial y})^2 - (\frac{\partial\phi}{\partial z})^2 - (\frac{\partial A_x}{c\partial t})^2 + (\frac{\partial A_x}{\partial y})^2 + (\frac{\partial A_x}{\partial z})^2 \qquad -\mathfrak{L}$$

   $$- (\frac{\partial A_y}{c\partial t})^2 + (\frac{\partial A_y}{\partial x})^2 + (\frac{\partial A_y}{\partial z})^2 - (\frac{\partial A_z}{c\partial t})^2 + (\frac{\partial A_z}{\partial x})^2 + (\frac{\partial A_z}{\partial y})^2) \qquad -\mathfrak{L}$$

   $$- \frac{\partial\phi}{\partial x}\frac{\partial A_x}{c\partial t} - \frac{\partial\phi}{\partial y}\frac{\partial A_y}{c\partial t} - \frac{\partial\phi}{\partial z}\frac{\partial A_z}{c\partial t} - \frac{\partial A_x}{\partial y}\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial z}\frac{\partial A_z}{\partial x} - \frac{\partial A_y}{\partial z}\frac{\partial A_z}{\partial y} \qquad -\mathfrak{L}$$

   $$+ \frac{\partial\phi}{\partial x}\frac{\partial A_x}{c\partial t} + (\frac{\partial\phi}{\partial x})^2 + \frac{\partial\phi}{\partial y}\frac{\partial A_y}{c\partial t} + (\frac{\partial\phi}{\partial y})^2 + \frac{\partial\phi}{\partial z}\frac{\partial A_z}{c\partial t} + (\frac{\partial\phi}{\partial z})^2 \qquad -\vec{\nabla}\cdot\phi\vec{E}$$

   $$= \frac{1}{2}(\frac{\partial A_x}{c\partial t})^2 + \frac{\partial\phi}{\partial x}\frac{\partial A_x}{c\partial t} + \frac{1}{2}(\frac{\partial\phi}{\partial x})^2 + \ldots y, z\,\text{terms}\ldots.$$

   $$+ \frac{1}{2}(\frac{\partial A_y}{\partial z})^2 - \frac{\partial A_y}{\partial z}\frac{\partial A_z}{\partial y} + \frac{1}{2}(\frac{\partial A_z}{\partial y})^2 + \ldots y, z\,\text{terms}\ldots$$

   $$= \frac{1}{2c^2}(\vec{E}^2 + \vec{B}^2).$$

Determine the momentum density along one coordinate.

1. Start with the Hamiltonian density:

$$h = \frac{\partial \mathfrak{L}}{\partial(\frac{\partial A^\lambda}{c\partial t})} \frac{\partial A^\lambda}{c\partial t} - \mathfrak{L} \; .$$

2. Generalize to make it a manifestly covariant second rank stress-energy tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = \frac{\partial \mathfrak{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - g^{\mu\nu}\mathfrak{L} \; .$$

3. Focus on one off-diagonal term:

$$T^{01} = -\frac{\partial \mathfrak{L}}{\partial(\frac{\partial A^\lambda}{c\partial t})} \frac{\partial A^\lambda}{\partial x} - g^{01}\mathfrak{L} \; .$$

4. Contract:

$$T^{01} = -\frac{\partial A_x}{c\partial t}\frac{\partial A_x}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{c\partial t}\frac{\partial A_y}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{c\partial t}\frac{\partial A_z}{\partial x} - \frac{\partial \phi}{\partial z}\frac{\partial A_z}{\partial x} \; .$$

5. Subtract $\vec{\nabla} \cdot A_i \vec{E}$, a factor need to make $T^{\mu\nu}$ symmetric:

$$\begin{aligned}
T^{01} = & -\frac{\partial A_x}{c\partial t}\frac{\partial A_x}{\partial x} - \frac{\partial \phi}{\partial x}\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{c\partial t}\frac{\partial A_y}{\partial x} - \frac{\partial \phi}{\partial y}\frac{\partial A_y}{\partial x} - \frac{\partial A_z}{c\partial t}\frac{\partial A_z}{\partial x} - \frac{\partial \phi}{\partial z}\frac{\partial A_z}{\partial x} \\
& + \frac{\partial A_x}{c\partial t}\frac{\partial A_x}{\partial x} + \frac{\partial \phi}{\partial x}\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{c\partial t}\frac{\partial A_x}{\partial y} + \frac{\partial \phi}{\partial y}\frac{\partial A_x}{\partial y} + \frac{\partial A_z}{c\partial t}\frac{\partial A_x}{\partial z} + \frac{\partial \phi}{\partial z}\frac{\partial A_x}{\partial z} \\
= & -\frac{\partial A_y}{c\partial t}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial \phi}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial A_z}{c\partial t}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) - \frac{\partial \phi}{\partial z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) \\
= & -\left(\frac{\partial A_y}{c\partial t} + \frac{\partial \phi}{\partial y}\right)\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \left(\frac{\partial A_z}{c\partial t} + \frac{\partial \phi}{\partial z}\right)\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) \\
= & E_y B_z - E_z B_y \qquad \Rightarrow \qquad T^{0i} = \vec{E} \times \vec{B} \; .
\end{aligned}$$