# Analytic Animations Using Quaternions

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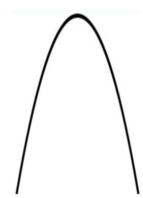
## Executive Summary

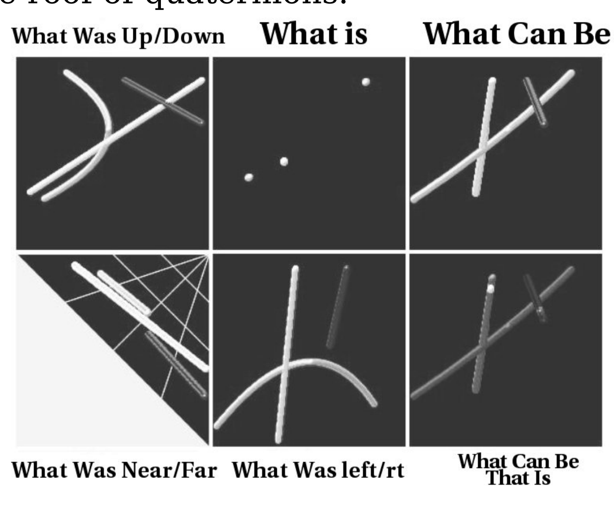
Quaternions are numbers with four parts, one for time, three for space. Analytic animations using quaternions generate animations directly from quaternion algebraic equations. Norms, sines, and cosines, all look new because these functions are now about time's relationship to space. A new perspective on quantum mechanics may be possible.

## Background



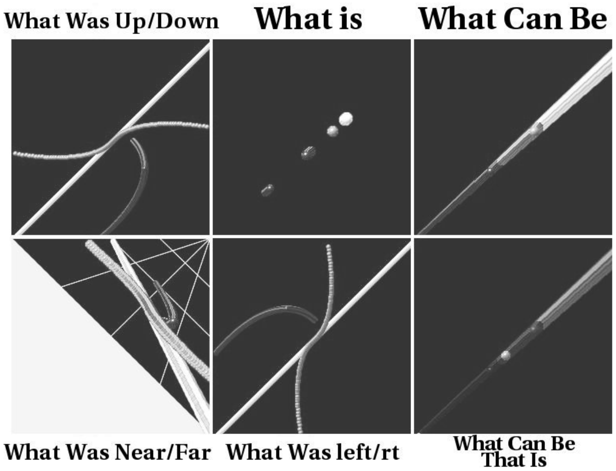
René Descarte is credited with the development of analytic geometry, the exact match between an algebraic equation and a drawing on a piece of paper. We know that mathematical reasoning for algebra happens in a different part of the brain than that for visual thinking, which is why the expression for a parabola $(x-a)^2-4by=0$ feels different from its corresponding graph. Once trained in these arts, it would be possible to recognize that the graph should be flipped over. Everything about one representation must be in the other. The simplicity of the graph suggest there should be a simple algebraic equation. The two approaches compliment and reinforce each other.

A real number has one number that can be added, subtracted, multiplied or divided by another to generate another real number (excluding division by zero). Quaternions are four numbers that also undergo these basic math operations to generate another quaternion. Quaternions, first noticed by Gauss, were rediscovered independently by Hamilton and Rodrigues. A small but vocal crew claimed quaternions were critical to understanding Nature. Gibbs took out what was useful: scalars, vectors, div, grad and curl. Now most students of math do not know that all those terms were born under one roof of quaternions.

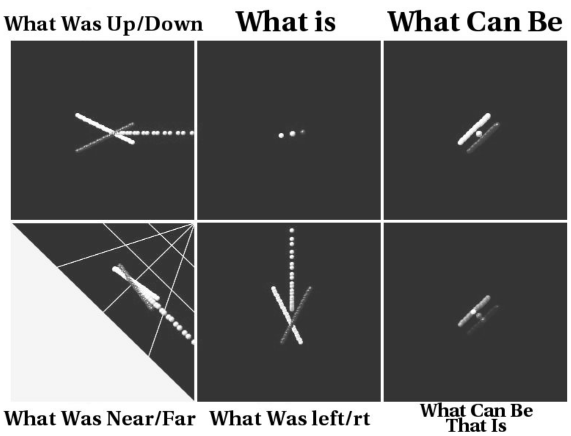



In the fierce, personal battles waged over the value of quaternions, no one could answer a simple question: what does it mean to multiply two quaternions together? A hundred and sixty years too late, I can answer that question. The first issue to address is what does it mean to add the same quaternion to another quaternion, over and over again. The result is known in physics as an inertial observer, but the simple answer is that it is a straight line animation. Take two straight line animations, and for each and every point, multiply the pairs together. The result is a curve ball through spacetime.

An essential innovation has been in information design.  Our visual systems record almost nothing.  Instead we are precise comparison shoppers.  The animations by themselves (“What is”) are able to hide information directly in front of me.  I surround the animation with still images. Three are the complex planes of tx, ty, and tz.  On the left is the superposition of all states. The curves through the complex planes of space/time appear sharper than those through space alone.




Sine and cosine are the workhorses of analytic geometry.  We all know what to expect, an unending series of camel humps from here to eternity. That is not how it looks for analytic animations. All the points line up straight. The provocative curves arise from time twisting with space. The input data is the light straight line.  Cosine in dark gray flies in at a right angle.  Sine in light gray rubs the input data at the origin, then heads off.  The curves that appear in the three “What was” graphs (complex planes), all aligned on the right.




What is the size of things?  In the world of vectors, the answer is a number, the norm, a solitary positivist, isolated from others. In an animation, that number is the length of time from the space origin 0, 0, 0.  The norm never moves left or right, the still Buddha, beating on the drum to say only how long it would take to go that far. The input stream in light gray crosses its sword with its conjugate in dark gray, and out fires the norm, at least in the three complex planes.  The animation is subtler, the light and dark gray lines always mirroring each other.  In a Universe with an equal amount of past as well as future, the norm shows up only in the second half.

## State of the Art

Quaternions are familiar to SIGGRAPH participants as the best tool for understanding 3D spatial rotations.  Perhaps the best book on the topic is “Visualizing Quaternions” by Andrew Hanson.  He has taught a course on quaternions at previous SIGGRAPH conferences.  The book starts out simple, but progresses to the most sophisticated levels of math.  It gets to the Riemann curvature tensor, at the core of Einstein's explanation of gravity, general relativity. The book has led to new analytic animations. For example, I have animations of the groups S0, S1, S2, S3, SU(2) and U(1)xSU(2), the last group having the symmetry of the electroweak force.

My major innovation was a starting assumption, that quaternions represent events in a spacetime manifold. Hanson treats 4D as a hypersphere, something that can be viewed a slice at a time. I consider 3D graphs to be quaternions, 3D in space and 1D in time, even if they never change in time.

A quaternion is the union of three complex numbers that share the same real number. In the animation representation, the real number is time, the three imaginary vectors are space. I intend to work through Tristan Needham's “Visual Complex Analysis” for algebra to animate. Simple things look deep. For the complex plane, reflection over the real axis looks the same as reflection over the imaginary axis. The story is different in an animation. The reflection over an imaginary axis looks like a mirror has done its job. For reflections of time, you need to remember what the path of the quaternion animation was like, then see it appear in reverse.

Since the command line has been the only stable interface that has lasted thirty years, I have chosen the command line as my interface for the quaternion library of functions written in C. Most animations start with a simple function, constant_linear_motion, which will generate any number of quaternions. These get fed into various combinations of unary or binary functions, using the flexibility of the command line. A Perl program takes the stream and connects it with various graphics programs. As of this date, the images can be generated by either Gnuplot or ImageMagick. I am planning on implementing drawings by POVRay, since that program does 3D depth very well.

The output is short animated GIFs and a corresponding web page. This facilitates getting the results up on the web where the reach is widest.

## A Tough Goal

Some equations don't look like animating them would be that interesting, like this one with a pair of exponentials: $q = A\exp(wt - KR) + B\exp(wt + KR)$ If all the constants are set to 1, the animation is a dot blinking at the origin. The reason it is important is its context: this is the plane wave solution for a free electromagnetic wave in a vacuum. It can be found in most books on quantum mechanics, such as Brandt and Dahmen's “The Picture Book of Quantum Mechanics”. Since this basic equation can be animated, it may be possible to animate any equation that appears in that book. This is the potential of analytic animations: there is necessarily an animation with the same information content as any equation in quantum mechanics. Very few people can understand the algebra on display in a graduate level quantum field theory book. The animations will be fun to look at no matter what a person's background is.

## Credits

All the physics research (http://quaternions.com/) and analytic animation work (http://quaternions.sf.net/) was done independent of any research institution or business by Doug Sweetser.